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An Impossible Superposition of Electromagnetic Waves in General Relativity

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This paper presents an example, similar to but distinct from one given in 1975 by Griffiths, of an insoluble problem on the superposition of a pair of electromagnetic waves.

1. INTRODUCTION

Some time ago Griffiths (1975) considered the problem of superimposing two independent electromagnetic waves propagating in different directions, and concluded that it was impossible for the two waves to follow simultaneously affinely parametrized shear-free and twist-free null geodesic congruences. The present paper deals with a similar problem, the point of difference from Griffiths being that instead of assuming affine parametrization, $\varepsilon=\gamma=0$, we set the spin coefficients α and β to zero. After all, these latter spin coefficients seem to be the least significant in terms of their immediate geometric meaning, and one might ask: could we not gain the simplest mathematical description of a given physical situation by scaling the basis spinors o^A, ι^A (or, equivalently, selecting a null tetrad based on a given pair of null directions) in such a way that $\alpha=\beta=0$? This is not the case, however. We find, in particular, that there can be no superposition of waves under the circumstances contemplated.

We will follow, as did Griffiths, the formalism of Newman and Penrose (1962). Once again, reference should be made to their paper for definition of notation, and equations quoted from it will be prefixed by NP.

Consider the superposition of a pair of electromagnetic waves, or null electromagnetic fields. If the basis spinors o^A, ι^A (or, equivalently, the basis null vectors l_μ, n_μ) are chosen along the repeated principal null directions of the two waves, the resultant electromagnetic field will satisfy

$$\Phi_0 \neq 0, \qquad \Phi_1 = 0, \qquad \Phi_2 \neq 0 \tag{1.1}$$

We assume, with Griffiths, that both waves are independently solutions of Maxwell's equations (NP A1). This leads to

$$\begin{aligned} D\Phi_2 &= (\rho - 2\varepsilon)\Phi_2 \\ \delta\Phi_2 &= (\tau - 2\beta)\Phi_2 \\ \Delta\Phi_0 &= (-\mu + 2\gamma)\Phi_0 \\ \bar{\delta}\Phi_0 &= (-\pi + 2\alpha)\Phi_0 \end{aligned} \tag{1.2}$$

and

$$\kappa = \sigma = \nu = \lambda = 0$$

The latter result means, of course, that the congruences of o^A and ι^A (or, equivalently, l_μ and n_μ) are geodesic and shear free.

By appropriately rescaling the basis spinors o^A and ι^A we now introduce, as mentioned above, the condition

$$\alpha = \beta = 0$$

For such a step to be possible, however, it must be assumed that

$$\rho = \bar{\rho}, \qquad \mu = \bar{\mu}, \qquad \Psi_2 = \mu\rho$$

This is explained in the Appendix.

The reality of ρ and μ, i.e., the absence of twist, is thus not an arbitrarily imposed restriction, as with Griffiths, but a necessary one. The opposite can be said of the final condition which we will now introduce, namely,

$$\tau + \bar{\pi} = 0$$

a condition invariant under transformations of scale of o^A, ι^A.

We now have a close counterpart to Griffiths' problem. Subsequent calculations will, however, of necessity take their own particular course.

2. BASIC EQUATIONS

The field equations (NP 4.2) reduce, in vacuum, under the conditions of the previous section, to the set

$$\begin{gathered}
\Psi_0=0, \qquad \Psi_2=\mu\rho, \qquad \Psi_4=0 \\
D\rho=\rho^2+(\varepsilon+\bar{\varepsilon})\rho+\Phi_{00} \\
\Delta\mu=-\mu^2-(\gamma+\bar{\gamma})\mu-\Phi_{22} \\
D\mu+\delta\bar{\tau}=2\mu\rho+\tau\bar{\tau}-(\varepsilon+\bar{\varepsilon})\mu \\
\Delta\rho-\bar{\delta}\tau=-2\mu\rho-\tau\bar{\tau}+(\gamma+\bar{\gamma})\rho \\
\delta\rho=-\Psi_1, \qquad \bar{\delta}\mu=\Psi_3 \\
D\tau=(\varepsilon-\bar{\varepsilon})\tau+\Psi_1 \\
\Delta\tau=(\gamma-\bar{\gamma})\tau+\bar{\Psi}_3 \\
\delta\tau=\tau^2+\Phi_{02}, \qquad \delta\varepsilon=\varepsilon\tau-\Psi_1 \\
\bar{\delta}\varepsilon=(\varepsilon+\rho)\bar{\tau}, \qquad \delta\gamma=(\gamma+\mu)\tau, \qquad \bar{\delta}\gamma=\gamma\bar{\tau}+\Psi_3 \\
D\gamma-\Delta\varepsilon=\mu\rho-\tau\bar{\tau}-(\varepsilon+\bar{\varepsilon})\gamma-(\gamma+\bar{\gamma})\varepsilon
\end{gathered} \tag{2.1}$$

With these and Maxwell's equations, the Bianchi identities (NP A3) become

$$\begin{aligned}
D\Psi_1&=(2\varepsilon+4\rho)\Psi_1-\bar{\Phi}_0\delta\Phi_0 \\
\Delta\Psi_1&=(2\gamma-3\mu)\Psi_1+\rho\bar{\Psi}_3-3\mu\rho\tau-\bar{\tau}\Phi_{02} \\
\delta\Psi_1&=4\tau\Psi_1+\bar{\Phi}_2D\Phi_0-2\varepsilon\Phi_{02} \\
\bar{\delta}\Psi_1&=2\bar{\tau}\Psi_1+\rho D\mu-2\mu\rho^2+(\varepsilon+\bar{\varepsilon})\mu\rho \\
D\Psi_3&=(3\rho-2\varepsilon)\Psi_3-\mu\bar{\Psi}_1-3\mu\rho\bar{\tau}-\tau\Phi_{20} \\
\Delta\Psi_3&=-(2\gamma+4\mu)\Psi_3-\bar{\Phi}_2\bar{\delta}\Phi_2 \\
\delta\Psi_3&=2\tau\Psi_3+\mu\Delta\rho+2\mu^2\rho-(\gamma+\bar{\gamma})\mu\rho \\
\bar{\delta}\Psi_3&=4\bar{\tau}\Psi_3+\bar{\Phi}_0\Delta\Phi_2+2\gamma\Phi_{20}
\end{aligned} \tag{2.2}$$

The commutators (NP 4.4) read

$$\Delta D - D\Delta = (\gamma + \bar{\gamma})D + (\varepsilon + \bar{\varepsilon})\Delta \tag{2.3a}$$

$$\delta D - D\delta = \tau D - (\rho + \varepsilon - \bar{\varepsilon})\delta \tag{2.3b}$$

$$\delta\Delta - \Delta\delta = \tau\Delta + (\mu - \gamma + \bar{\gamma})\delta \tag{2.3c}$$

$$\bar{\delta}\delta - \delta\bar{\delta} = 0 \tag{2.3d}$$

Maxwell's equations (1.2) also, of course, belong among what we may call "basic equations."

3. SOME PRELIMINARY RESULTS

Let us observe, first of all, that none of the three spin coefficients μ, ρ, τ can vanish, for otherwise condition (1.1) would be violated, by virtue of the field equations (2.1).

We now apply to Ψ_1 the commutator (2.3a), and replace derivatives where possible by substituting from equations (1.2), (2.1), or (2.2). A term containing $\Delta\delta\Phi_0$ is expanded by using the commutator (2.3c). Equivalently we could begin by applying to Ψ_1 the commutator (2.3d) instead of (2.3a). In either case we are eventually left with

$$0 = \Psi_1[3\mu(\varepsilon + \bar{\varepsilon}) - 4\rho(\gamma + \bar{\gamma}) + 2\mu\rho + 2\tau\bar{\tau} + 3D\mu + 4\Delta\rho]$$
$$+ \Phi_{02}\left[\bar{\Psi}_1 - 2\bar{\tau}(\varepsilon + \rho)\right] + 3\mu\rho\tau(\varepsilon + \bar{\varepsilon} - 2\rho) + 3\rho\tau D\mu + \bar{\tau}\bar{\Phi}_2 D\Phi_0 \tag{3.1}$$

Repeating for Ψ_3, i.e., applying either the commutator (2.3a) or the commutator (2.3d), we similarly get

$$0 = \Psi_3[-3\rho(\gamma + \bar{\gamma}) + 4\mu(\varepsilon + \bar{\varepsilon}) - 2\mu\rho - 2\tau\bar{\tau} + 3\Delta\rho + 4D\mu]$$
$$+ \Phi_{20}\left[-\bar{\Psi}_3 - 2\tau(\gamma + \mu)\right] + 3\mu\rho\bar{\tau}(\gamma + \bar{\gamma} - 2\mu) - 3\mu\bar{\tau}\Delta\rho - \tau\bar{\Phi}_0\Delta\Phi_2 \tag{3.2}$$

Next we apply the commutator (2.3d) to μ and ρ, obtaining, respectively,

$$\tau\Psi_3 = \bar{\tau}\bar{\Psi}_3 \tag{3.3}$$

and

$$\bar{\tau}\Psi_1 = \tau\overline{\Psi}_1 \tag{3.4}$$

whence we see that $\Psi_1\Psi_3$ is real.

Operating on relations (3.3) and (3.4) with δ, we get, respectively,

$$\begin{aligned}\bar{\tau}\Phi_0\Delta\overline{\Phi}_2 = &\left(\tau\mu - \overline{\Psi}_3\right)\Delta\rho + \Phi_{02}\Psi_3 - \rho(2\mu - \gamma - \bar{\gamma})\overline{\Psi}_3 \\ &+ \mu\rho\tau(2\mu - \gamma - \bar{\gamma}) - 2\tau\bar{\tau}\overline{\Psi}_3 - 2\bar{\gamma}\bar{\tau}\Phi_{02}\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\bar{\tau}\overline{\Phi}_2 D\Phi_0 = &(\tau\rho + \Psi_1)D\mu + \Phi_{02}\overline{\Psi}_1 - \mu(2\rho - \varepsilon - \bar{\varepsilon})\Psi_1 \\ &- \mu\rho\tau(2\rho - \varepsilon - \bar{\varepsilon}) - 2\tau\bar{\tau}\Psi_1 + 2\varepsilon\bar{\tau}\Phi_{02}\end{aligned} \tag{3.6}$$

while the operator D, applied to relation (3.3), gives

$$\tau^2\Phi_{20} = \bar{\tau}^2\Phi_{02} \tag{3.7}$$

Finally, for this section, we eliminate the term $\bar{\tau}\overline{\Phi}_2 D\Phi_0$ between relations (3.1) and (3.6), and likewise the term $\bar{\tau}\Phi_0\Delta\overline{\Phi}_2$ between the complex conjugate of (3.2) and (3.5). Taking note that

$$D\mu + \Delta\rho = (\gamma + \bar{\gamma})\rho - (\varepsilon + \bar{\varepsilon})\mu$$

which immediately follows from two of the field equations (2.1), the respective results are

$$2\rho\tau D\mu = 2\mu\rho\tau(2\rho - \varepsilon - \bar{\varepsilon}) - \Phi_{02}\left(\overline{\Psi}_1 - \rho\bar{\tau}\right) \tag{3.8}$$

and

$$2\mu\tau\Delta\rho = -2\mu\rho\tau(2\mu - \gamma - \bar{\gamma}) - \Phi_{02}(\Psi_3 + \mu\bar{\tau}) \tag{3.9}$$

We see, incidentally, that

$$\mu\Psi_1 + \rho\overline{\Psi}_3 = 0$$

4. COMPLETION OF THE PROOF

It is convenient to define two real quantities H and ξ by setting

$$H=\mu(2\rho-\varepsilon-\bar{\varepsilon})-D\mu \tag{4.1}$$

$$=\rho(2\mu-\gamma-\bar{\gamma})+\Delta\rho$$

and

$$\xi=2\tau\bar{\tau}H\left(\tau^{2}\Phi_{20}\right)^{-1} \tag{4.2}$$

Results (3.8) and (3.9) then become

$$\Psi_{1}=\rho\tau(1+\xi), \qquad \Psi_{3}=-\mu\bar{\tau}(1+\xi) \tag{4.3}$$

We now operate on the relation (4.1) with δ, using the commutator (2.3b) to replace $\delta D\mu$, and using equations (2.1) and (2.2). The result is

$$\delta H=\tau H+\bar{\tau}\Phi_{02} \tag{4.4}$$

Next we operate with δ on the relation (4.2), using field equations (2.1) (notice that $\delta\bar{\tau}=H+\tau\bar{\tau}$), the result (4.4), and the fact that $\delta\Phi_{20}=2\tau\Phi_{20}$, which is an immediate consequence of the electromagnetic equations (1.2) with $\alpha=\beta=0$, $\pi=-\bar{\tau}$. Relation (3.7) is also involved here, enabling us to express Φ_{02} in terms of Φ_{20}. In this way we obtain

$$\bar{\tau}\delta\xi=2(\tau\bar{\tau}-H)-(\tau\bar{\tau}-H)\xi \tag{4.5}$$

Thirdly we operate with δ on the second of the two relations (4.3) [or with $\bar{\delta}$ on the first of (4.3)], using the result (4.5). After canceling out a factor ξ [$\xi\neq 0$, for otherwise also $H=0$ and then, by (4.4), $\bar{\tau}\Phi_{02}=0$, which is disallowed], we are left with

$$\xi=2H(\tau\bar{\tau})^{-1}-4 \tag{4.6}$$

Finally we operate with δ on this last relation. Using several of our earlier results, we arrive at the following cubic equation for the real quantity $H(\tau\bar{\tau})^{-1}$, which we now denote for simplicity by P:

$$P^{3}-3P^{2}+4P-3=0 \tag{4.7}$$

This of course means that P is a constant.

We now infer from (4.6) that ξ is constant, and then from (4.5) with (4.6) that

$$(P-1)(P-3)=0$$

Neither of the two roots of this quadratic satisfies the equation (4.7). We have thus arrived at a contradiction. There can be no superposition of electromagnetic waves under the assumed conditions.

5. COMMENT ON $\alpha=\beta=\varepsilon=\gamma=0$

We might generalize and ask, what kind of solutions of the field equations are possible, given that the basis spinors o^A, ι^A (or, equivalently, the basis null vectors l_μ, n_μ) follow a particular pair of null directions, and $\alpha, \beta, \varepsilon, \gamma$ all vanish. The possibility of setting the four spin coefficients to zero depends on the existence of a complex scalar Z such that (cf. the Appendix)

$$DZ=-Z\varepsilon, \qquad \Delta Z=-Z\gamma$$

$$\delta Z=-Z\beta, \qquad \bar{\delta}Z=-Z\alpha$$

These equations in turn imply, via the commutators (NP 4.4) and their complex conjugates, a total of six relations among the spin coefficients and components of Ψ and Φ which can be found more quickly by simply setting $\alpha=\beta=\varepsilon=\gamma=0$ in the field equations (NP 4.2). The six relations are a serious restriction on the variety of possible solutions.

Thus there will be no superposition of electromagnetic fields which are independent solutions of Maxwell's equations (NP A1) under the condition (1.1), even if the condition $\tau+\bar{\pi}=0$ is removed. In the absence of an electromagnetic field, only particular choices of gravitational fields are possible (we will not burden this paper with details). It is interesting to note, however, that there does exist an electrovac solution on record with $\alpha=\beta=\varepsilon=\gamma=0$, namely, that of Tariq and Tupper (1975). It refers to an algebraically general electromagnetic field which is not a superposition of two null fields.

APPENDIX

We have stated that the possibility of setting $\alpha=\beta=0$ requires that $\rho=\bar{\rho}$ and $\mu=\bar{\mu}$. This can be seen by considering the effect on α and β of the transformation $o^A \to Zo^A$, $\iota^A \to Z^{-1}\iota^A$ of the basis spinors, where Z is some complex scalar. We find

$$\alpha = \iota^B \iota^D o^{\dot{X}} \nabla_{D\dot{X}} o_B \to Z^{-2}\bar{Z}(Z\alpha + \bar{\delta}Z)$$

$$\beta = \iota^B o^D \iota^{\dot{X}} \nabla_{D\dot{X}} o_B \to \bar{Z}^{-1}(Z\beta + \delta Z)$$

In order to transform α and β to zero, we thus require the existence of Z such that

$$\delta Z = -Z\beta, \qquad \bar{\delta}Z = -Z\alpha$$

For integrability, however, the commutators (NP 4.4) must be satisfied. From the last of the commutators we now get, also using the field equation (NP 4.21), that

$$Z(\mu\rho - \Psi_2) + (\rho - \bar{\rho})(Z\gamma + \Delta Z) + (\mu - \bar{\mu})(Z\varepsilon + DZ) = 0$$

which is satisfied for any DZ and ΔZ, provided that

$$\rho = \bar{\rho}, \qquad \mu = \bar{\mu}, \qquad \Psi_2 = \mu\rho$$

REFERENCES

Griffiths, J. B. (1975). *International Journal of Theoretical Physics*, **13**, 9.

Newman, E., and Penrose, R. (1962). *Journal of Mathematical Physics*, **3**, 566; (1963). *Ibid.*, **4**, 998.

Tariq, N., and Tupper, B. O. J. (1975). *General Relativity and Gravitation*, **6**, 345.